A finite soluble quotient algorithm

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Abstract

An algorithm computing power conjugate presentations for finite soluble quotients of finitely presented groups is described. An implementation of this algorithm is available.

1 Introduction

Polycyclic groups are characterised by the fact that they have a polycyclic series, which is a descending series of subgroups, such that each one is normal in the previous one and their quotient is cyclic (see Segal, 1983, or for computational aspects Sims, 1994). Polycyclic presentations describe polycyclic groups by exhibiting a polycyclic series of the group. They are very important for computing in the group since they allow the practical computation (by collection) of a normal word for every element in the group. Algorithms using such descriptions for finite polycyclic groups are, for example, described in Laue et al. (1984) and form an integral part of the computational group theory systems Cayley (Cannon, 1984) and GAP (Schönert et al., 1993).

Every polycyclic group is soluble and every finite soluble group is polycyclic. However not every soluble group is polycyclic. Baumslag, Cannonito, and Miller (1981a, 1981b) describe an algorithm which decides whether a soluble group given by a finite presentation is polycyclic. It has been partly implemented by Sims (1990). Here attention is focused on finite soluble groups. Polycyclic presentations for finite soluble groups are better known as power conjugate presentations.

The task of a finite soluble quotient algorithm is to compute power conjugate presentations for finite soluble groups described as quotients of finitely presented groups. A number of proposals for finite soluble quotient algorithms have been made, for instance by Wamsley (1977), by Leedham-Green (1984) and by Plesken (1987). The last algorithm has been developed, analysed and implemented by Wegner (1992).

Algorithms for computing power conjugate presentations for p-groups or for nilpotent groups described as quotients of finitely presented groups exist. For p-groups see for example Havas and Newman (1980) or Celler et al. (1993) and for nilpotent groups see Sims (1994) or Nickel (in preparation).

Here a new finite soluble quotient algorithm is presented in detail. For a brief description of the algorithm see Niemeyer (to appear). New features are the use of vector enumeration and the intermediate presentations considered.

We now turn to the background required for the description of the algorithm. Let G be a finite soluble group and let $G = G_0 \geq G_1 \geq \cdots \geq G_n = \{1\}$ be a composition series for G with factors of prime order. Choose elements $a_i \in G$ for $1 \leq i \leq n$ such that $G_{i-1} = \langle G_i, a_i \rangle$; let p_i be the order of the factor G_{i-1}/G_i. Then $\mathcal{A} = \{a_1, \ldots, a_n\}$ is a generating set for G. Choose words v_{jk} in the elements $a_{j+1}, \ldots, a_n$ for $1 \leq j \leq k \leq n$ such that $a_i^{p_i} = v_{ii}$ for $1 \leq i \leq n$ and $a_k^{a_j} = v_{jk}$ for $1 \leq j < k \leq n$ and let $\mathcal{R}$ be the set consisting of these relations. Then $\mathcal{R}$ is a defining set of relations for G. The presentation $\{\mathcal{A} \mid \mathcal{R}\}$ is a *power conjugate presentation* for G. A power conjugate presentation $\{\mathcal{A} \mid \mathcal{R}\}$ of a group G *exhibits* the composition series $G = G_0 \geq G_1 \geq \cdots \geq G_n = \langle 1 \rangle$, where $G_{i-1} = \langle a_i, \ldots, a_n \rangle$ for $1 \leq i \leq n$. The order of G is at most $\prod_{i=1}^n p_i$ and therefore G is finite. A word $w(a_1, \ldots, a_n)$ in the generators is *normal* if it is of the form $a_1^{e_1} \cdots a_n^{e_n}$ with $0 \leq e_i < p_i$. Note that we only consider words in the elements of $\mathcal{A}$, that is these words do not contain inverses of the generators.

In what follows "word" means semigroup word. A normal word u in $\mathcal{A}$ is *equivalent* to a word w in $\mathcal{A}$ if u and w are the same element of the group defined by $\{\mathcal{A} \mid \mathcal{R}\}$. The fundamental importance of power conjugate presentations arises from the observation that, given a word w in the generators $\mathcal{A}$, the power conjugate presentation $\{\mathcal{A} \mid \mathcal{R}\}$ can be used to compute an equivalent normal word. It is assumed that the words v_{jk} in a power conjugate presentation are normal. If the right hand side of some relation is the identity then the relation is written as a relator by just listing the left hand side.

If a word is not normal it has a non-normal subword minimal in the partially ordered set of non-normal words. This subword is of the form a_i^p or of the form $a_j a_i$ with $j > i$. *Collection* of a word consists of a sequence of steps each of which chooses a minimal non-normal subword and replaces it. A subword of the form a_i^p is replaced by v_{ii} and $a_k^{a_j}$ is replaced by $a_k v_{jk}$ (see also Celler et al., 1993). For normal words the sequence of collection steps is empty. Otherwise a minimal non-normal subword is chosen and replaced. Each further step is applied to the result of the previous step. The words resulting from these steps are equivalent to the given word. For an account of collection see for example Havas and Nicholson (1976), or Leedham-Green and Soicher (1990).

A fundamental result is that every collection (independent of the choices of minimal non-normal subwords) of a non-normal word results in a normal word after a finite number of steps (see for example Sims, 1994). A normal word resulting from collecting the word w will be denoted (w); it may depend upon the choices made in the process. Multiplication of two elements of G amounts to computing a normal word for the product given by concatenation.

In general, there may exist many normal words representing a given group element. If each element is represented by a unique normal word, then the power conjugate presentation is *consistent*. In this case two group elements are equal only if they are represented by the same normal word. For the finite soluble group G, given as above, the order is then equal to $\prod_{i=1}^n p_i$.

The following result is due to Wamsley (1977). In summary it states that a power conjugate presentation is consistent if certain words, called *consistency test words*, can be collected in "sufficiently different" ways and still yield the same normal word. These consistency test words are $a_k\,a_j a_i$ with $1 \leq i < j < k \leq n$, $a_k^p a_j$, with $1 \leq j < k \leq n$, $a_j a_i^p$, with $1 \leq i < j \leq n$, and a_i^{p+1} with $1 \leq i \leq n$.

Theorem 1 *Let G be a finite soluble group given by the power conjugate presentation $\{\mathcal{A} \mid \mathcal{R}\}$. Then the presentation $\{\mathcal{A} \mid \mathcal{R}\}$ is consistent if and only if the following equations hold:*

$$
\begin{aligned}
\big((a_k\,a_j)\,a_i\big) &= \big(a_k\,(a_j\,a_i)\big) && \textit{for} \quad 1 \leq i < j < k \leq n, \\
\big((a_k^p)\,a_j\big) &= \big(a_k^{p-1}\,(a_k\,a_j)\big) && \textit{for} \quad 1 \leq j < k \leq n, \\
\big((a_j\,a_i)\,a_i^{p-1}\big) &= \big(a_j\,(a_i^p)\big) && \textit{for} \quad 1 \leq i < j \leq n, \\
\big((a_i^p)\,a_i\big) &= \big(a_i\,(a_i^p)\big) && \textit{for} \quad 1 \leq i \leq n.
\end{aligned}
$$

In practice we are interested in describing groups by consistent power conjugate presentations. In this context power conjugate presentations which exhibit a refinement of a specific series of a given group are considered. The algorithm described here can be viewed as a generalisation of the prime quotient algorithm described by Havas and Newman (1980) and by Celler et al. (1993). Let G be a group and p a prime. The prime quotient algorithm works with a series

$$G = \mathcal{P}_0^p(G) \geq \mathcal{P}_1^p(G) \geq \cdots \qquad \text{with } \mathcal{P}_i^p(G) = [\mathcal{P}_{i-1}^p(G), G]\big(\mathcal{P}_{i-1}^p(G)\big)^p \text{ for } i \geq 1$$

called the *lower exponent-p central series* of G. If there exists an integer $c \geq 0$ such that $\mathcal{P}_c^p(G) = \langle 1 \rangle$, then G is a p-group and the smallest such integer is called the *exponent-p class* of G. It repeats a basic step which, given a consistent power conjugate presentation of $G/\mathcal{P}_i(G)$, computes a consistent power conjugate presentation of $G/\mathcal{P}_{i+1}(G)$.

In the next section a series is defined that takes the role of the lower exponent-p central series in the context of finite soluble groups. The power conjugate presentations exhibiting this series are described.

2 The soluble $\mathcal{L}$-series

Let G be a group. Let $\mathcal{L} = [(p_1, c_1), \ldots, (p_k, c_k)]$ be a list of pairs consisting of a prime, p_i, and a non-negative integer, c_i, with $p_i \neq p_{i+1}$ and c_i positive for $i < k$. For $1 \leq i \leq k$ and $0 \leq j \leq c_i$ define the list $\mathcal{L}_{i,j} = [(p_1, c_1), \ldots, (p_{i-1}, c_{i-1}), (p_i, j)]$. Set $\mathcal{L}_{1,0}(G) = G$. For $1 \leq i \leq k$ and $1 \leq j \leq c_i$ let

$$\mathcal{L}_{i,j}(G) = \mathcal{P}_j^{p_i}(\mathcal{L}_{i,0}(G))$$

and for $1 \le i < k$ let

$$\mathcal{L}_{i+1,0}(G) = \mathcal{L}_{i,c_i}(G)$$

and $\mathcal{L}(G) = \mathcal{L}_{k,c_k}(G)$. Note that $\mathcal{L}_{i,j}(G) \ge \mathcal{L}_{i,j+1}(G)$ holds for $j < c_i$.

The chain of subgroups

$$G = \mathcal{L}_{1,0}(G) \ge \mathcal{L}_{1,1}(G) \ge \cdots \ge \mathcal{L}_{1,c_1}(G) = \mathcal{L}_{2,0}(G) \ge \cdots \ge \mathcal{L}_{k,c_k}(G) = \mathcal{L}(G)$$

is called the *$\mathcal{L}$-series* of G. If $\mathcal{L}(G) = \langle 1 \rangle$ then G is an *$\mathcal{L}$-group.* If $c_k > 0$ and $\tilde{\mathcal{L}}(G) \ne \langle 1 \rangle$ for $\tilde{\mathcal{L}} = [(p_1, c_1), \ldots, (p_k, c_k - 1)]$ then G is a *strict $\mathcal{L}$-group.*

Note that in the definition of strict $\mathcal{L}$-group the exponent-p_k class of $\mathcal{L}_{k,0}(G)$ is determined but not necessarily the exponent-p_i class of $\mathcal{L}_{i,0}(G)$. For every finite soluble group G there exists a (not necessarily unique) list $\mathcal{L}$ such that G is a strict $\mathcal{L}$-group.

For a given i the series $\mathcal{L}_{i,0}(G) \ge \cdots \ge \mathcal{L}_{i,c_i}(G)$ is an initial segment of the lower exponent-p_i central series of $\mathcal{L}_{i,0}(G)$ and $\mathcal{L}_{i,0}(G)/\mathcal{L}_{i,c_i}(G)$ is a p_i-group of exponent-p_i class at most c_i.

We use the following notation.

$$\mathcal{L}^{+p} = \begin{cases} [(p_1, c_1), \ldots, (p_k, c_k + 1)] & \text{if } p = p_k, \\ [(p_1, c_1), \ldots, (p_k, c_k), (p, 1)] & \text{if } p \ne p_k. \end{cases}$$

Then $\mathcal{L}(G)/\mathcal{L}^{+p}(G)$ is an elementary abelian p-group. Further

$$\mathcal{L}^{-p} = \begin{cases} [(p_1, c_1), \ldots, (p_{k-1}, c_{k-1})] & \text{if } p = p_k, \\ [(p_1, c_1), \ldots, (p_k, c_k)] & \text{if } p \ne p_k. \end{cases}$$

Let $\{\mathcal{A} \mid \mathcal{R}\}$ be a power conjugate presentation for a finite soluble group H with $\mathcal{A} = \{a_1, \ldots, a_n\}$. Let d be the minimal number of generators in $\mathcal{A}$ required to generate H. Assume there exists a d-element subset $\mathcal{X}$ of $\mathcal{A}$ such that $\mathcal{X}$ generates H and for each generator $a \in \mathcal{A} \backslash \mathcal{X}$ there is at least one relation of $\mathcal{R}$ having a as the *last* generator on the right hand side and occurring with exponent 1. Choose exactly one of these relations and call it the *definition* of a. The fact that this relation is the definition of a is emphasised by using '=:' instead of '=' in the relation. The colon is on the same side of the relation as the generator defined by this relation. The presentation $\{\mathcal{A} \mid \mathcal{R}\}$ together with chosen definitions for the generator in $\mathcal{A} \backslash \mathcal{X}$ is called *labelled.* Let G be a group with generating set $\{g_1, \ldots, g_b\}$ and τ an epimorphism of G onto H. For $i = 1, \ldots, b$ let w_i be the normal word equivalent to $\tau(g_i)$. If $a \in \mathcal{X}$ is the *last* generator in at least one w_i occurring with exponent 1 and we have chosen one such w_i, we write $\tau(g_i) =: w_i$ and call this the *definition of a.* For each $a \in \mathcal{X}$ there is a maximal k and a maximal c such that $a \in \mathcal{L}_{k,c}(H)$. We call τ a *labelled* epimorphism if each generator $a \in \mathcal{X}$ has a definition $\tau(g_i) =: w_i$ and a is the only generator which occurs in w_i and does lie in $\mathcal{L}_{k,c}(H)$. If $\{\mathcal{A} \mid \mathcal{R}\}$ is a labelled power conjugate presentation for the group H and τ a labelled epimorphism from G to H, then every generator in $\mathcal{A}$ has a definition either as an image under τ or as a relation in $\mathcal{R}$. Further we can read off a preimage in G for each $a \in \mathcal{A}$ under τ. Thus we can compute a preimage in H for each $g \in G$ under τ. The cardinality of $\mathcal{X}$ is called the *generator number* of G with respect to $\{\mathcal{A} \mid \mathcal{R}\}$. Even though the generator number depends on the power conjugate presentation the reference to the presentation is often omitted.

The following is a labelled consistent power conjugate presentation for S_4, the symmetric group on 4 letters:

$$
\begin{array}{lllll}
\{ a,\, b,\, c,\, d \mid & a^2 =: c, & & & \\
& b^a = b^2 c, & b^3, & & \\
& c^a = c, & c^b =: d, & c^2, & \\
& d^a = cd, & d^b = cd, & d^c = d, & d^2 \, \}.
\end{array}
$$

The relations $a^2 = c$ and $c^b = d$ are the definitions of c and d, respectively. Note that this notation implicitly characterises the set $\mathcal{X}$ as the subset of $\mathcal{A}$ whose elements do not occur as the last element of a right hand side of a definition. In the example, $\mathcal{X}$ is the set $\{a, b\}$.

Consider the group G having the finite presentation

$$\{x,\, y \mid x^8,\, y^3,\, (x^{-1}y)^2,\, (yx^3yx)^2 = x^4\}.$$

Then the map τ from G to S_4 defined by $\tau(x) =: a$ and $\tau(y) =: b$ is a labelled epimorphism. The elements of $\mathcal{X}$ have definitions as images of τ, whereas the other elements in $\mathcal{A}$ have definitions in the relations of the power conjugate presentation for S_4.

3 The $\mathcal{L}$-covering group

Let $\mathcal{L}$ be the list $[(p_1, c_1), \ldots, (p_k, c_k)]$, where c_i is a non-negative integer for $1 \le i \le k$, and let K be an $\mathcal{L}$-group with generator number d. Let F be the free group of rank d and let θ be an epimorphism of F onto K. Let p be a prime and denote $\mathcal{L}^{-p}(K)$ by P and let F_P be the preimage in F of P under θ.

Let K be a finite strict $\mathcal{L}$-group with generator number d. A group H is a *p-descendant* of K if H has generator number d and H is an $\tilde{\mathcal{L}}$-group, where for some non-negative integer n

$$\tilde{\mathcal{L}} = \begin{cases} [(p_1, c_1), \ldots, (p_k, c_k + n)], & \text{if } p = p_k, \\ [(p_1, c_1), \ldots, (p_k, c_k), (p, n)] & \text{if } p \neq p_k. \end{cases}$$

and $\tilde{\mathcal{L}}^{-p}(H)$ is isomorphic to K. If H is a strict $\mathcal{L}^{+p}$ group, that is $n = 1$, then H is an *immediate p-descendant* of K.

Note that if K is a p-group these definitions are the same as in O'Brien (1990).

Theorem 2 *Let K be a finite strict $\mathcal{L}$-group with generator number d and p a prime. There exists an $\mathcal{L}^{+p}$-group $\hat{K}$ with generator number d such that every immediate p-descendant of K is isomorphic to a quotient of $\hat{K}$.*

Proof: Let F be the free group of rank d and let R be the kernel of the epimorphism θ of F onto K. Let F_P be the preimage under θ in F of $P = \mathcal{L}^{-p}(K)$. Define S to be $[R, F_P]R^p$ and define the group $\hat{K}$ to be F/S. Then $S \leq R$ and $\hat{K}$ is a d-generator group. Let H be an immediate p-descendant of K and let ν be an epimorphism of H onto K. Then there exists an epimorphism $\hat{\theta}$ from F to H such that $\hat{\theta}\nu = \theta$. Since $R\hat{\theta} \leq \ker \nu$ it follows that $R\hat{\theta}$ is an elementary abelian p-subgroup of H, which is central in $F_P\hat{\theta}$. Hence $S\hat{\theta} = [R\hat{\theta}, F_P\hat{\theta}](R\hat{\theta})^p$ is the identity in H and thus H is a homomorphic image of F/S. ∎

If p is not p_k then P is the identity and F_P is R. Therefore $S = [R, R]R^p$ and R/S is the relation module of F/R (see Gruenberg, 1976).

The group $\hat{K} = F/S$ with $S = [R, F_P]R^p$ is the *$\mathcal{L}$-covering group of K with respect to the prime p*. A consistent power conjugate presentation $\{\hat{\mathcal{A}} \mid \hat{\mathcal{R}}\}$ for $\hat{K}$ is an *$\mathcal{L}$-covering presentation* of $\{\mathcal{A} \mid \mathcal{R}\}$.

It should always be clear from the context which prime p is chosen. Therefore $\hat{K}$ is called the $\mathcal{L}$-covering group of K without reference to p. If $\mathcal{L}$ is the list $[(p, c)]$ then an $\mathcal{L}$-group K is a p-group and the $\mathcal{L}$-covering group $\hat{K}$ is the p-covering group K^* of K. O'Brien (1990) shows that K^* is isomorphic to $F/[R, F]R^p$.

The $\mathcal{L}$-covering group F/S is the largest extension of F/R by an elementary abelian p-group such that the extension has the same generator number as F/R and F_P/R acts trivially on the elementary abelian p-group. Note that the generator number of F_P/S may be larger than the generator number of F_P/R. The p-covering group $(F_P/R)^*$ of F_P/R is the largest extension of F_P/R by an elementary abelian p-group which has the same generator number as F_P/R and F_P/R acts trivially on the elementary abelian p-group.

The following theorem asserts that the isomorphism type of F/S is independent of the choice of the homomorphism θ from F to K and thereby independent of its kernel R. It is valid also for the case that $\mathcal{L}^{-p}(F/R)$ is trivial.

Theorem 3 *Let R_1 and R_2 be normal subgroups of F such that F/R_1 and F/R_2 are $\mathcal{L}$-groups. Furthermore, let U_1 and U_2 be subgroups of F such that for $i \in \{1, 2\}$*

1) $R_i \leq U_i$,

2) $U_i/R_i \leq \mathcal{L}^{-p}(F_P/R_i)$,

3) U_i/R_i *is characteristic in* F/R_i,

and there exists an isomorphism φ *of* F/R_1 *onto* F/R_2, *which maps* U_1/R_1 *onto* U_2/R_2. *Then* $F/[R_1, U_1]R_1^p$ *is isomorphic to* $F/[R_2, U_2]R_2^p$ *by an isomorphism which takes* R_1/S_1 *to* R_2/S_2 *and* U_1/S_1 *to* U_2/S_2.

Proof: Let $\{a_1, \ldots, a_d\}$ be a free generating set for F. Define S_i to be $[R_i, U_i]R_i^p$ for $i \in \{1, 2\}$. Let ν be the canonical epimorphism of F/S_1 onto F/R_1. Then $\nu\varphi$ is an epimorphism from F/S_1 onto F/R_2. Let $b_i \in F$ such that $(b_iS_1)\nu\varphi = a_iR_2$. Define a homomorphism $\rho : F \to F/S_1$ mapping a_i to b_iS_1. The map $\rho\nu\varphi$ is an epimorphism from F onto F/R_2. Since $\rho\nu\varphi$ agrees on each a_i with the natural projection of F onto F/R_2 and since the a_i generate F, $\rho\nu\varphi$ is the natural projection. Thus $R_2\rho\nu\varphi = 1$, and $R_2\rho \leq \ker \nu\varphi = R_1/S_1$ and $U_2\rho\nu\varphi = U_2/R_2$, so $U_2\rho \leq (U_2/R_2)\varphi^{-1}\nu^{-1} = (U_1/R_1)\varphi^{-1} = U_1/S_1$. It follows that $S_2\rho = ([R_2, U_2]{R_2}^p)\rho$ is a subgroup of $[R_1/S_1, U_1/S_1](R_1/S_1)^p$. Since the elements of R_1/S_1 are of order p and commute with U_1/S_1, we have that $S_2 \leq \ker \rho$. Hence F/S_1 is isomorphic to a factor group of F/S_2. Similarly F/S_2 is isomorphic to a factor group of F/S_1, and therefore $F/S_1 \cong F/S_2$. ∎

If the subgroup U_i is chosen to be $\mathcal{L}^{-p}(F_p/R_i)$ then the theorem asserts that given a d-generator $\mathcal{L}$-group G with $P = \mathcal{L}^{-p}(G)$, the choice of the epimorphism $\theta : F \to G$ and thus the choice of $R = \ker \theta$ does not influence the isomorphism type of F/S. We can also choose $U_i = R_i$ and thus the choice of the epimorphism θ also has no impact on the isomorphism type of $F/[R, R]R^p$.

4 An $\mathcal{L}$-covering algorithm

The task of the $\mathcal{L}$-covering algorithm is to determine a labelled consistent power conjugate presentation for the $\mathcal{L}$-covering group of a soluble $\mathcal{L}$-group K.

- The input of the $\mathcal{L}$-covering algorithm is a labelled consistent power conjugate presentation for a soluble $\mathcal{L}$-group K and a prime p.
- The output is a consistent power conjugate presentation for the soluble group $\hat{K}$, the $\mathcal{L}$-covering group of K with respect to the prime p.

The $\mathcal{L}$-covering algorithm presented here first computes a finite presentation for $\hat{K}$. It is shown that one can define a normal form for the elements in $\hat{K}$ and that the finite presentation can be used like a power conjugate presentation to compute normal forms of elements of $\hat{K}$. Applying a theorem which is a generalisation of Theorem 1 allows the determination of a module presentation for the kernel of the natural epimorphism of $\hat{K}$ onto K. A vector space basis for this $\mathbb{F}_p K$-module is computed by an algorithm, called vector enumeration, and this in turn enables the determination of a labelled consistent power conjugate presentation for $\hat{K}$.

The individual steps of the algorithm are illustrated by reference to the example of the symmetric group on 4 letters, S_4. A consistent power conjugate presentation for this group was given in Section 2. For this example let $\mathcal{L}$ be the list $[(2,1),(3,1),(2,1)]$ and let p be the prime 2.

4.1 A finite presentation for the $\mathcal{L}$-covering group

A finite presentation $\{\tilde{\mathcal{A}} \mid \tilde{\mathcal{R}}\}$ for $\hat{K} = F/S$ is obtained in the following way. Let $\{\mathcal{A} \mid \mathcal{R}\}$ be the labelled consistent power conjugate presentation for the $\mathcal{L}$-group K, where $\mathcal{A}$ is the set $\{a_1, \ldots, a_n\}$ and

$$\mathcal{R} = \{a_i^{p_i} = v_{ii},\ a_k^{a_j} = v_{jk} \mid 1 \le i \le n,\ 1 \le j < k \le n\}.$$

Then $\{\mathcal{A} \mid \mathcal{R}\}$ is a power conjugate presentation for K with respect to a composition series which refines the soluble $\mathcal{L}$-series. Therefore there exists an r such that $\{a_1P, \ldots, a_rP\}$ generates K/P and $\{a_{r+1}, \ldots, a_n\}$ generates P. Let s be the number of relations in $\mathcal{R}$ which are not definitions. Then $s = (n-1)n/2 + d$. Introduce new generators $\{y_1, \ldots, y_s\}$ and define $\tilde{\mathcal{A}} = \{a_1, \ldots, a_n\} \cup \{y_1, \ldots, y_s\}$. We obtain $\tilde{\mathcal{R}}$ in the following way:

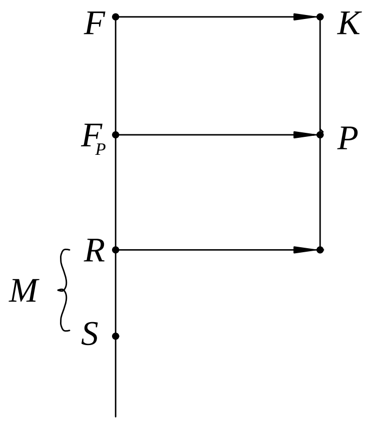
1) initialise $\tilde{\mathcal{R}}$ to contain all relations of $\mathcal{R}$ which are definitions;
2) modify each non-defining relation $a_i^{p_i} = v_{ii}$ or $a_k^{a_j} = v_{jk}$ of $\mathcal{R}$ to read $a_i^{p_i} = v_{ii}y_t$ or $a_k^{a_j} = v_{jk}y_t$ for some $t \in \{1, \ldots, s\}$, where different non-defining relations are modified by different y_t, and add the modified relation to $\tilde{\mathcal{R}}$;
3) add to $\tilde{R}$ all relations of the form $[y_i, y_j^g] = 1$ for all normal $g = w(a_1, \ldots, a_r)$ and $y_i^p = 1$ for $1 \le i, j \le s$;
4) add to $\tilde{\mathcal{R}}$ all relations $y_i^{a_j} = y_i$ for $j > r$ and $1 \le i \le s$.

We apply this to the example of S_4. The subgroup $\mathcal{L}_{3,0}(S_4)$ is a 2-group isomorphic to the Klein 4-group and is generated by c and d. Further, S_4 is generated by a and b. The definition of c is the relation with left-hand side a^2 and the definition of d is the relation with left-hand side c^b. We obtain the following presentation for $\hat{S}_4$:

$\tilde{\mathcal{A}} = \{a, b, c, d, y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8\}$ and

$$\begin{aligned}
\tilde{\mathcal{R}} = \{a^2 &= c, \\
b^a &= b^2cy_1, \quad b^3 = y_2, \\
c^a &= cy_3, \quad c^b = d, \quad c^2 = y_4, \\
d^a &= cdy_5, \quad d^b = cdy_6, \quad d^c = dy_7, \quad d^2 = y_8, \\
y_i^c &= y_i, \quad \text{for } 1 \le i \le 8, \\
y_i^d &= y_i, \quad \text{for } 1 \le i \le 8, \\
[y_i, y_j^g] &= 1 \quad \text{for } 1 \le i, j \le 8 \text{ and } g \in \{a, ab, b\}\}.
\end{aligned}$$

Consider the following diagram.



The subgroup R/S, denoted by M, is the kernel of the natural epimorphism of $\hat{K}$ to K and can be characterised as follows. It is the maximal $\mathbb{F}_p K$-module by which K can be extended so that P acts trivially on M and the extension has the same generator number as K. Thus M is an $\mathbb{F}_p(K/P)$-module. Let Y be the free $\mathbb{F}_p(K/P)$-module on $\{y_1, \ldots, y_s\}$. The module M is a homomorphic image of Y. The kernel of the homomorphism from Y onto M can be computed effectively. In order to see this, we study the finite presentation for the group $\hat{K}$ in more detail.

4.2 Collecting in $\hat{K}$

One can collect in the group $\hat{K}$ relative to $\{\tilde{\mathcal{A}} \mid \tilde{\mathcal{R}}\}$. The definition of a normal word can be generalised for this presentation in the following way. A word in $\tilde{\mathcal{A}}$ is *normal* if it is of the form $w(a_1, \ldots, a_n) \cdot \Pi_{i=1}^{s} y_i^{f_i}$, where $w(a_1, \ldots, a_n)$ is a normal word in $\{a_1, \ldots, a_n\}$ and f_i is an element of $\mathbb{F}_p(K/P)$. The following steps, referred to as "collection in $\hat{K}$", can be applied to every word in $\tilde{\mathcal{A}}$. For $f, f' \in \mathbb{F}_p(K/P)$ and $1 \leq k, l \leq s$ and $1 \leq i, j \leq n$

1) replace $y_l^f y_k^{f'}$ by $y_k^{f'} y_l^f$ for $k < l$;
2) replace $y_k^f y_k^{f'}$ by $y_k^{f+f'}$;
3a) replace $y_k^f a_i^q$ by $a_i y_k^{(f a_i)} a_i^{q-1}$ if $q > 1$;
3b) replace $y_k^f a_i$ by $a_i y_k^{(f a_i)}$;
4) replace $a_i^{p_i}$ by v, where $a_i^{p_i} = v$ is a relation in $\tilde{\mathcal{R}}$;
5) replace $a_j a_i$ by $a_i v$, where $a_j^{a_i} = v$ is a relation in $\tilde{\mathcal{R}}$ for $i < j$.

In each step a word is replaced by another word representing the same element of $\hat{K}$. After applying a finite number of these steps to any word it is replaced by a normal word. This can be proved in a way similar to proving that a collection process computes a normal word after applying finitely many collection steps. Rules 1), 2) and 3) use the fact that M is an $\mathbb{F}_p(K/P)$-module. Note that 4) and

5) resemble collection steps in a collection algorithm, where the power conjugate presentation is used to determine the replacement.

For example the word $b(ba)$ in collects in $\hat{S}_4$ to $abdy_1^{b^2+1}y_2^b y_4 y_6 y_7$.

The following lemma states that two equivalent normal words can differ only by a module word in Y.

Lemma 4 *Let w be an arbitrary word in $\{a_1, \ldots, a_n\} \cup \{y_1, \ldots, y_s\}$. Then there exists a unique normal word v in $\{a_1, \ldots, a_n\}$ such that any normal word in $\hat{K}$ equivalent to w has the form $v \cdot \Pi_{i=1}^{s} y_i^{f_i}$.*

Proof: The existence of the unique normal word v follows from the fact that $\{\mathcal{A} \mid \mathcal{R}\}$ is a consistent power conjugate presentation for K. ∎

A consequence of this lemma is that the map $\phi : \hat{K} \to K$ which maps a word in $\hat{K}$ to its unique normal word in $\{a_1, \ldots, a_n\}$ is an epimorphism.

The following theorem allows us to describe the kernel of the homomorphism from Y onto $M = R/S$ in a manner suitable for computation. It considers certain non-normal words in $\hat{K}$.

Theorem 5 *Let Y be the free $\mathbb{F}_p(K/P)$-module on $\{y_1, \ldots, y_s\}$ and $\{\tilde{\mathcal{A}} \mid \tilde{\mathcal{R}}\}$ the presentation for the extension $\hat{K}$ of K as defined above. Let W be the following set of consistency test words in $\{a_1, \ldots, a_n\}$:*

$$\begin{array}{ll}
\left(\left((a_k\, a_j)\, a_i\right) \left(a_k\, (a_j\, a_i)\right)^{-1}\right) & \text{for} \quad 1 \le i < j < k \le n, \\
\left(\left((a_k^p)\, a_j\right) \left(a_k^{p-1}\, (a_k\, a_j)\right)^{-1}\right) & \text{for} \quad 1 \le j < k \le n, \\
\left(\left((a_j\, a_i)\, a_i^{p-1}\right) \left(a_j\, (a_i^p)\right)^{-1}\right) & \text{for} \quad 1 \le i < j \le n, \\
\left(\left((a_i^p)\, a_i\right) \left(a_i\, (a_i^p)\right)^{-1}\right) & \text{for} \quad 1 \le i \le n
\end{array}$$

and let T be the set of elements obtained by collecting the words in W with respect to $\{\tilde{\mathcal{A}} \mid \tilde{\mathcal{R}}\}$. Then T consists of words in Y and M is isomorphic to $Y/(T\,\mathbb{F}_p(K/P))$.

Proof: The elements of W represent the identity element in $\hat{K}$. By Lemma 4 the elements of T are words in Y. Denote $T\,\mathbb{F}_p(K/P)$ by $\langle T \rangle$. Let $\mu : Y \to M$ be the epimorphism mapping y_i in Y to y_i in M. Since the elements of W are the identity in $\hat{K}$ it follows that the elements of T, when viewed as elements of Y, are mapped to the identity element of M, hence $\langle T \rangle \subseteq \ker \mu$. Therefore $Y/\langle T \rangle$ has a factor module isomorphic to M.

We now define a consistent power conjugate presentation for K extended by $Y/\langle T \rangle$ such that the extension is a d-generator group. Since M is the largest $\mathbb{F}_p(K/P)$-module with these properties it follows that $Y/\langle T \rangle$ is isomorphic to M.

Let $\{b_1, \ldots, b_m\}$ be a vector space basis for $Y/\langle T\rangle$. Then each element $y_i\langle T\rangle$ can be expressed uniquely in the basis elements. Therefore we obtain a power conjugate presentation $\{\hat{\mathcal{A}} \mid \hat{\mathcal{R}}\}$ for an extension of K by $Y/\langle T\rangle$ in the following way from the presentation $\{\tilde{\mathcal{A}} \mid \tilde{\mathcal{R}}\}$ where $\hat{\mathcal{A}}$ is $\mathcal{A} \cup \{b_1, \ldots, b_m\}$:

1) replace every occurrence of an element y_i on the right hand side of a relation in $\tilde{\mathcal{R}}$ with left hand side a word in $\mathcal{A}$, by the corresponding word for $y_i\langle T\rangle$ in the basis and add this modified relation to $\hat{\mathcal{R}}$;
2) add to $\hat{\mathcal{R}}$ the relations $b_i^{a_j} = w_i(b_1, \ldots, b_m)$ for all $1 \leq i \leq m$ and $1 \leq j \leq r$, where $w_i(b_1, \ldots, b_m)$ is determined by the action of a_j on b_i;
3) add to $\hat{\mathcal{R}}$ the relations $b_i^{a_j} = b_i$ for $1 \leq i \leq m$ and $r+1 \leq j \leq n$;
4) add to $\hat{\mathcal{R}}$ the relations $b_j^{b_i} = b_j$ for $1 \leq i < j \leq m$ and the relations $b_i^p = 1$ for $1 \leq i \leq m$.

We now show that $\{\hat{\mathcal{A}} \mid \hat{\mathcal{R}}\}$ is consistent. Let $\tilde{K}$ denote the group defined by $\{\hat{\mathcal{A}} \mid \hat{\mathcal{R}}\}$. Since the elements of W collect to elements of $\langle T\rangle$ they are the trivial word in $\tilde{K}$.

The consistency of $\{\hat{\mathcal{A}} \mid \hat{\mathcal{R}}\}$ is proved by applying Theorem 1. We only need to consider consistency relations which involve at least one element of the basis. Any consistency relation which involves only basis elements holds, since the basis is a basis for a vector space over $\mathbb{F}_p$.

Consider the word $b_k a_j a_i$. Applying a collection step to $(b_k a_j) a_i$ with respect to $\{\hat{\mathcal{A}} \mid \hat{\mathcal{R}}\}$ yields $a_j b_k^{a_j} a_i$ which collects to $a_j a_i (b_k^{a_j})^{a_i}$ and finally to $a_i v_{ij} (b_k^{a_j a_i})$. On the other hand $b_k(a_j a_i)$ collects to $b_k a_i v_{ij}$ which in turn collects to $a_i v_{ij} b_k^{(a_i v_{ij})}$. Since $(b_k^{a_j a_i})$ and $b_k^{(a_i v_{ij})}$ are the same module element they have the same normal form in the basis.

Consider the word $b_k b_j a_i$. Applying a collection step to $(b_k b_j) a_i$ with respect to $\{\hat{\mathcal{A}} \mid \hat{\mathcal{R}}\}$ yields $b_j b_k a_i$ which collects to $a_i b_j^{a_i} b_k^{a_i}$. On the other hand $b_k(b_j a_i)$ collects to $b_k a_i b_j^{a_i}$ which collects to $a_i b_k^{a_i} b_j^{a_i}$. Therefore the first consistency relation in Theorem 1 holds. Similarly one can prove that the other relations also hold and therefore $\{\hat{\mathcal{A}} \mid \hat{\mathcal{R}}\}$ is a consistent power conjugate presentation. ∎

In collecting the words in W with respect to the presentation $\{\tilde{\mathcal{A}} \mid \tilde{\mathcal{R}}\}$ we obtain a set T which generates the kernel of the epimorphism of the free $\mathbb{F}_p(K/P)$-module Y onto the module M.

In the example T is the set

$$
\begin{array}{ll}
\{\; y_2, & y_1^{(1+b+b^2)} y_3 y_5 y_6 y_7, \\
\;\; y_3^{(a+1)}, & y_3 y_6 y_7^{(a+1)}, \\
\;\; y_3^{b} y_7, & y_3 y_6 y_7^{(1+b)}, \\
\;\; y_1^{(1+a+b^2)} y_2 y_3^{(1+b^2)} y_5^{(1+b+b^2)} y_6^{(1+b^2)} y_7, & \\
\;\; y_2^{(a+1)}, & y_2 y_3 y_4^{(a+1)} y_6, \\
\;\; y_5^{(1+b)} y_6 y_7, & y_5^{(b+b^2)} y_6^{b} y_7^{b}, \\
\;\; y_4 y_5 y_6, & y_2 y_3 y_4^{(1+b^2)} y_5^{(a+b)} y_6^{(1+b)} y_7^{b}, \\
\;\; y_6^{(1+a)}, & y_6^{(1+b)} \qquad\qquad \}.
\end{array}
$$

In the proof of the previous theorem it was assumed that we have

- a vector space basis $\{b_1, \ldots, b_m\}$ for the $\mathbb{F}_p(K/P)$-module $Y/\langle T \rangle$;
- an expression in the basis for $b_i^{a_j}$ for $1 \leq i \leq m$ and $1 \leq j \leq r$;
- an expression in the basis for $y_i\langle T \rangle$ for $1 \leq i \leq s$.

Where such information is available the proof yields a constructive method to obtain a consistent power conjugate presentation $\{\hat{\mathcal{A}} \mid \hat{\mathcal{R}}\}$ for $\hat{K}$. We now describe an algorithm which may be used to obtain this information.

4.3 Computing a vector space basis for a module

The technique of vector enumeration is used to compute a basis for the $\mathbb{F}_p(K/P)$-module M needed to obtain a power conjugate presentation for F/S. A vector enumeration algorithm is described in Linton (1991). Its use in this context has been suggested by Leedham-Green (private communication, 1991).

It is used with the following input:

1) a consistent power conjugate presentation for K;
2) the set of free generators for Y;
3) a set T.

The output is:

1) an $\mathbb{F}_p$-basis $\{b_1, \ldots, b_m\}$ for M;
2) the matrix action of each generator of K/P in the power conjugate presentation of K/P on M with respect to the computed basis;
3) an expression in the computed basis for $y_i\langle T \rangle$ for $1 \leq i \leq s$.

This output is used to obtain a consistent power conjugate presentation for the extension $\hat{K}$ of K by M using the method described in the proof of Theorem 5.

We illustrate the technique by reference to our example. The vector enumerator with input $\{\mathcal{A} \mid \mathcal{R}\}$, the set $\{y_1, y_2, y_3, y_4, y_5, y_6, y_7\}$ and T as above computes a module basis for the module M. The basis has five elements $\{e, f, g, h, i\}$ defined by $e = y_1$, $f = y_2$, $g = y_3$, $h = e^a$ and $i = e^b$. Further the vector enumerator gives the action of a and b on the module basis, while the elements c and d act trivially.

The information returned by the vector enumerator can be used to construct the following consistent power conjugate presentation for the $\mathcal{L}$-covering group $\hat{S}_4$:

$$
\begin{array}{llllllll}
\{a,b,c,d,e,f,g,h,i,j \mid \\
a^2 = c, \\
b^a = b^2ce, \; b^3, \\
c^a = c, & c^b = d, & c^2 = f, \\
d^a = cdg, & d^b = cdh, & d^c = dgh, d^2 = i, \\
e^a = j, & e^b = j, & e^c = e, & e^d = e, \; e^2, \\
f^a = f, & f^b = i, & f^c = f, & f^d = f, f^e = f, f^2, \\
g^a = fh, & g^b = hi, & g^c = g, & g^d = g, \; g^e = g, \; g^f = g, \; g^2, \\
h^a = fg, & h^b = gi, & h^c = h, & h^d = h, h^e = h, h^f = h, h^g = h, h^2, \\
i^a = fghi, i^b = fghi, & & i^c = i, & i^d = i, \; i^e = i, \; i^f = i, \; i^g = i, \; i^h = i, \; i^2, \\
j^a = e, & j^b = efgij, j^c = j, & & j^d = j, \; j^e = j, \; j^f = j, \; j^g = j, \; j^h = j, j^i = j, j^2\}.
\end{array}
$$

From the presentation we can read off that $\hat{S}_4$ has order $2^9 3 = 1536$. The group $\mathcal{L}^{-2}(\hat{S}_4)$ has order 2^8 and is generated by $\{c,d,e,f,g,h,i,j\}$. It is the direct product of the normal subgroups $\langle c,d\rangle$ and $\langle e,j,fgi\rangle$ of $\hat{S}_4$. In general, the preimage $\hat{P}$ of $P = \mathcal{L}^{-p}(K)$ is a normal subgroup of $\mathcal{L}^{-p}(\hat{K})$. It contains a normal subgroup, namely $\hat{P} \cap M$. Note that r was defined such that the subset $\{a_{r+1},\dots,a_n\}$ of $\mathcal{A}$ generates P. A generating set for the normal subgroup $\hat{P}\cap M$ is the union of the set $\{v_{ii} \mid a^p = v_{ii}$ is a relation in $\hat{\mathcal{R}}$ for $i > r\}$ and the set $\{v_{ij} \mid a_i^{a_j} = a_i v_{ij}$ is a relation in $\hat{\mathcal{R}}$ for $i,j > r\}$. It can thus be obtained from the power conjugate presentation for $\hat{K}$.

The group ring $\mathbb{F}_p(K/P)$ in the example is isomorphic to $\mathbb{F}_2 S_3$. We can investigate the module structure of M as an S_3-module further. The submodule $\hat{P} \cap M$ is the direct sum of $\langle fi, ghi\rangle$ and $\langle gh\rangle$. Its module complement $\langle e,j,fgi\rangle$ is a direct sum of a one dimensional and a two dimensional module. It has the decomposition $\langle fgi\rangle \oplus \langle ej, efgi\rangle$.

4.4 Obtaining a labelled presentation

In some cases additional work is necessary to transform the consistent power conjugate presentation $\{\hat{\mathcal{A}} \mid \hat{\mathcal{R}}\}$ of $\hat{K}$ into a labelled presentation. It is possible that a basis vector b_i does not occur as the last element of the right hand side of a relation in $\hat{\mathcal{R}}$ and thus no relation can be chosen as the definition of b_i. In this case we proceed as follows. For each basis vector b_i choose a relation which contains b_i in its right hand side as the definition of b_i, ensuring that this relation is not chosen as the definition of any other basis vector. Assume that for the element b_i the right hand side of its defining relation has the form

$$w_1(a_1,\dots,a_n)\cdot w_2(b_1,\dots,b_{i-1})\cdot w_3(b_i,\dots,b_m).$$

Define the element $\tilde{b}_i$ to be $w_3(b_i, \ldots, b_m)$. Obviously $\{\tilde{b}_1, \ldots, \tilde{b}_m\}$ is again a vector space basis for M and the action of the generator a_j of K/P on this basis can be computed as the action of a_j on $w_3(b_i, \ldots, b_m)$ and then expressing the result in the new basis. A labelled consistent power conjugate presentation for $\hat{K}$ is obtained by performing a base change.

The power conjugate presentation for $\hat{S}_4$ given above is already a labelled power conjugate presentation, where c, d, e, f, g, h, i and j are defined by the relations with left hand sides a^2, c^b, b^a, c^2, d^a, d^b, d^2 and e^a, respectively. Every extension of S_4 by an elementary abelian 2-group M such that the Klein 4-group acts trivially on M and the extension has generator number 2 is isomorphic to a quotient of the group defined by this presentation.

5 A soluble quotient algorithm

The soluble quotient algorithm presented here computes a power conjugate presentation for a quotient $G/\mathcal{L}(G)$ of a finitely presented group G, where the presentation exhibits a composition series of the quotient group which is a refinement of the soluble $\mathcal{L}$-series. It takes as input:

1) a finite presentation $\{g_1, \ldots, g_b \mid r_1(g_1, \ldots, g_b), \ldots, r_m(g_1, \ldots, g_b)\}$ for G;
2) a list $\mathcal{L} = [(p_1, c_1), \ldots, (p_k, c_k)]$, where each p_i is a prime, $p_i \neq p_{i+1}$, and each c_i is a positive integer.

The output is:

1) a labelled power conjugate presentation for $G/\mathcal{L}(G)$ exhibiting a composition series refining the soluble $\mathcal{L}$-series of this quotient;
2) a labelled epimorphism $\tau : G \twoheadrightarrow G/\mathcal{L}(G)$.

The algorithm proceeds by computing power conjugate presentations for the quotients $G/\mathcal{L}_{i,j}(G)$ in turn. Without loss of generality assume that a power conjugate presentation for $G/\mathcal{L}_{i,j}(G)$ has been computed for $j < c_i$. The basic step computes a power conjugate presentation for $G/\mathcal{L}_{i,j+1}(G)$. The group $\mathcal{L}_{i,j}(G)/\mathcal{L}_{i,j+1}(G)$ is a p_i-group. The basic step takes as input:

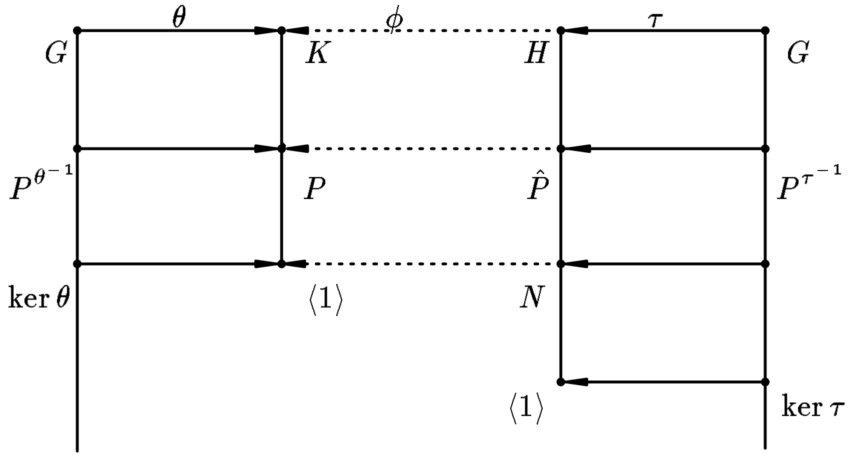
1) the finite presentation for G;
2) a labelled consistent power conjugate presentation for the finite soluble quotient $K \cong G/\mathcal{L}_{i,j}(G)$ of G with $j < c_i$ which refines the $\mathcal{L}$-series of K;
3) a labelled epimorphism $\theta : G \twoheadrightarrow K$.

The output is:

1) a labelled consistent power conjugate presentation for the finite soluble group $H \cong G/\mathcal{L}_{i,j+1}(G)$, exhibiting a composition series refining the $\mathcal{L}$-series of H;
2) an epimorphism $\phi : H \twoheadrightarrow K$;
3) a labelled epimorphism $\tau : G \twoheadrightarrow H$ with $\tau\phi = \theta$.

If during the basic step it is discovered that $\mathcal{L}_{i,j}(G) = \mathcal{L}_{i,j+1}(G)$, then $\mathcal{L}_{i+1,0}(G)$ is set to $\mathcal{L}_{i,j}(G)$.

The basic step is illustrated by the following diagram, where the input is described on the left and the output is described on the right. Put $p = p_i$, let P denote $\mathcal{L}_{i,0}(K)$, and $\hat{P}$ denote $\mathcal{L}_{i,0}(H)$. If $j = 0$ then P is trivial. The elementary abelian p-group $\ker\phi$ is denoted by N. The group $\hat{P}$ acts trivially on N; thus $\hat{P}$ is a central extension of P by N, and $\hat{P}$ is a p-group of exponent-p class at most one larger than the exponent-p class of P.



The subgroup $N = \ker\phi$ plays a role similar to that of the subgroup M in the $\mathcal{L}$-covering algorithm. It is the maximal $\mathbb{F}_p K$-module by which K can be extended so that P acts trivially on N and the extension is an epimorphic image of G. Thus N is an $\mathbb{F}_p(K/P)$-module. If P is non-trivial the extension of K by N has the same generator number as K, because, by Burnside's Basis Theorem (Huppert I, Satz 3.15, 1967), the generator number of the extension of P by N is already determined by the generator number of P. If P is non-trivial the module M is the largest $\mathbb{F}_p K$-module by which K can be extended such that the extension has the same generator number as K and P acts trivially on M. Therefore N is a factor module of M. If P is trivial and N is the largest $\mathbb{F}_p K$-module by which K can be extended such that the extension is a homomorphic image of G, it does not follow that N is isomorphic to a factor module of M, since the extension may have a larger generator number than K. However, in both cases we can write down a finite presentation for the extension.

This presentation is obtained as follows. Let $\{\mathcal{A} \mid \mathcal{R}\}$ be the supplied consistent power conjugate presentation for K, where $\mathcal{A} = \{a_1, \ldots, a_n\}$ and

$$\mathcal{R} = \{a_i^{p_i} = v_{ii},\ a_k^{a_j} = v_{jk} \mid 1 \le i \le n,\ 1 \le j < k \le n\}.$$

Let $\{\tilde{\mathcal{A}} \mid \tilde{\mathcal{R}}\}$ with $\tilde{\mathcal{A}} = \{a_1, \ldots, a_n, y_1, \ldots, y_s\}$ be the finite presentation for $\hat{K}$ as calculated by the $\mathcal{L}$-covering algorithm. Then $G/\mathcal{L}_{i,j+1}(G)$ is isomorphic to a quotient of $\hat{K}$, if P is nontrivial. If P is trivial, the generator number of $G/\mathcal{L}_{i,j+1}(G)$ may be larger than the generator number of $G/\mathcal{L}_{i,j}(G)$. Since $G/\mathcal{L}_{i,j+1}(G)$ has a consistent power conjugate presentation refining its $\mathcal{L}$-series it follows that any additional generators lie in $\mathcal{L}_{i,j}(G)/\mathcal{L}_{i,j+1}(G)$. Therefore N is isomorphic to a quotient of the direct product Z of the free $\mathbb{F}_p(K/P)$-module Y and the free $\mathbb{F}_p(K/P)$-module on the additional generators. Let t be the number of generators of G whose images under θ are not definitions, then $t = b - d$, where b is the number of generators of G in the finite presentation and d is the generator number of K. Add new generators $\{z_1, \ldots, z_t\}$ to $\tilde{\mathcal{A}}$. The set of relations $\tilde{\mathcal{R}}$ is modified in the following manner.

1) add to $\tilde{R}$ all relations of the form $[z_i, z_j^g] = 1$, $[y_k, z_j^g] = 1$ and $[z_i, y_k^g] = 1$ for all normal $g = w(a_1, \ldots, a_r)$ for $1 \leq i, j \leq t$ and $1 \leq k \leq m$ and all relations $z_i^p = 1$ for $1 \leq i \leq t$;
2) add to $\tilde{R}$ all relations $z_i^{a_j} = z_i$ for $j > r$ for $1 \leq i \leq t$.

The group $\tilde{K}$ defined by $\{\tilde{\mathcal{A}} \mid \tilde{\mathcal{R}}\}$ has $G/\mathcal{L}_{i,j+1}(G)$ as a factor group. It is called the *extended $\mathcal{L}$-covering group of K* and $\{\tilde{\mathcal{A}} \mid \tilde{\mathcal{R}}\}$ is the *extended $\mathcal{L}$-covering presentation.* Define a map σ from $\{g_1, \ldots, g_b\}$ to the group $\tilde{K}$ by $g_i^\sigma = g_i^\theta z_k$ if g_i^θ is non-defining and $g_i^\sigma = g_i^\theta$ if g_i^θ is defining. The map σ is called the *extended map*.

The basic step is illustrated by an example. Consider the group G defined by the following finite presentation $\{x, y \mid x^8, y^3, (x^{-1}y)^2, (yx^3yx)^2 = x^4\}$. Let $\mathcal{L}$ be the list $[(2,1),(3,1),(2,2)]$. Then it can be shown that $G/\mathcal{L}_{3,1}(G)$ is isomorphic to S_4. A labelled consistent power conjugate presentation for S_4 was given above. The input for the basic step is the finite presentation for G, the labelled consistent power conjugate presentation for S_4 and the epimorphism $\theta : G \to G/\mathcal{L}_{3,1}(G)$ defined by $x \mapsto a$ and $y \mapsto b$. The images of θ are the definitions of a and b, respectively. We have previously determined a presentation for $\hat{S}_4$. This is also the extended $\mathcal{L}$-covering presentation since both images of θ are definitions. The map σ is the map from G to $\tilde{K}$ which maps x to a and y to b. Using the map σ the kernel of the homomorphism from Z onto N can be computed effectively.

Theorem 6 *Let T be the set of elements in Theorem 5. Let U be the set $\{r_i(g_1^\sigma, \ldots, g_b^\sigma) \mid 1 \leq i \leq m\}$ of elements of Z obtained by evaluating the relators of G in the images of the generators of G under the map σ. Then N is isomorphic to $Z/((T \cup U)\mathbb{F}_p(K/P))$.*

Proof: Consider the factor group H of $\tilde{K}$ obtained by extending K by $Z/((T \cup U)\mathbb{F}_p(K/P))$. Then H is generated by $g_1^\sigma, \ldots, g_b^\sigma$. Since the relations of G hold in H it follows that it is a homomorphic image of $G/\mathcal{L}_{i,j}(G)$. By construction H has $G/\mathcal{L}_{i,j}(G)$ as a homomorphic image, hence H is isomorphic to $G/\mathcal{L}_{i,j}(G)$. ∎

In our example U is the set $\{y_1^b y_3^{(bab)} y_4^b y_5^b y_7^{(1+b)}\}$.

The vector enumerator was used to compute a vector space basis for the module M in the $\mathcal{L}$-covering algorithm. Here it is employed to compute a vector space basis for the module $N = Z/((T \cup U)\mathbb{F}_p(K/P))$. It takes as input

1) a consistent power-conjugate presentation for K;
2) the set of generators for Z;
3) the set of relations $T \cup U$.

The output is

1) an $\mathbb{F}_p$-basis for N;
2) an expression in this basis for the image under the generators of K/P of every basis element;
3) expressions for the images of the $\mathbb{F}_p(K/P)$-generators of Z in terms of the basis elements.

This output is used to obtain a consistent power conjugate presentation for the extension H of K by N, an epimorphism τ from G to H and an epimorphism ϕ from H to K. The method for constructing the consistent power conjugate presentation is again the method described in the proof of Theorem 5. The homomorphism τ from G to H is obtained by replacing the elements z_i in the map σ by the corresponding word in the basis for N. This yields an epimorphism by Theorem 6. As pointed out earlier a base change for the vector space basis of the module N may be necessary in order to obtain a labelled consistent power conjugate presentation for H. A base change may also be necessary in order to transform τ into a labelled homomorphism. The map τ is an epimorphism, since all the generators of H are either defined as images of the generators of G under τ or by definitions in the power conjugate presentation of H on the images of those generators.

The vector enumerator is employed to compute a vector space basis for the module $Z/((T \cup U)\mathbb{F}_2(K/P))$ in the previous example. The vector enumerator returns the basis $\{e, f, g\}$ defined by $e = y_3$, $f = y_4$ and $g = y_5$. Again, the vector enumerator gives the action of a and b on this basis, while c and d act trivially. The information is used to construct for the quotient $H = G/\mathcal{L}_{4,0}(G)$ the following labelled consistent power conjugate presentation:

$$
\begin{array}{l}
\{\, a,\, b,\, c,\, d,\, e,\, f,\, g \mid \\
\quad a^2 =: c, \\
\quad b^a = b^2 c,\; b^3, \\
\quad c^a = c,\quad c^b =: d,\quad c^2 =: e \\
\quad d^a =: cdf,\; d^b =: cdg,\; d^c = dfg,\; d^2 = ef \\
\quad e^a = e,\quad e^b = ef,\; e^c = e,\quad e^d = e,\; e^2 \\
\quad f^a = eg,\; f^b = efg,\; f^c = f,\quad f^d = f,\; f^e = f,\; f^2 \\
\quad g^a = ef,\; g^b = e,\quad g^c = g,\quad g^d = g,\; g^e = g,\; g^f = g,\; g^2 \,\}.
\end{array}
$$

and the labelled epimorphism τ from G onto $G/\mathcal{L}_{4,0}(G)$ defined by $x \mapsto a$ and $y \mapsto b$. Hence in this example G has a homomorphic image isomorphic to a factor group of $\hat{S}_4$. In fact G is an extension of S_4 by a group N of order 2^3. The group N is generated by $\{c^2, d^2, [d,c]\}$ and therefore $\mathcal{L}^{-2}(H)$ is isomorphic to the 2-covering group of the Klein-4 group.

ACKNOWLEDGEMENTS

I thank my PhD supervisor Dr M.F. Newman for his generous support and assistance; I also thank Dr L.G. Kovács, Dr C.R. Leedham-Green, Dr Werner Nickel and Dr E.A. O'Brien for many encouraging discussions and generous help. I acknowledge the support of an OPRSA and an ANU PhD scholarship during which this work was carried out. Part of the writing was supported by ARC Grant A69230241.

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